My research is at the intersection of commutative algebra and representation theory. Within commutative algebra, I am interested in the structure theory of finite free resolutions and the closely related study of perfect ideals. The theory of linkage has been applied in this field to great effect, and in §1 we survey some of the important results from the literature.

Representation theory enters the picture from two surprising directions. The first is that certain Schubert (ind-)varieties carry exactly the right symmetry to study linkage. This perspective allows us to unify the results of §1 and it makes explicit predictions regarding the behavior of perfect ideals in more complicated cases, as we explain in §2.

The second is that the same objects in representation theory also naturally appear in the algebraic construction of certain generic free resolutions, and this is briefly discussed in §3. This connection was discovered by Weyman, and most of my work has focused on analyzing it. This circle of ideas can be summarized as follows:

$$\begin{tikzcd}[column sep=huge] \text{free resolutions and perfect ideals} \arrow[rr, dash, "\text{generic free resolutions}", "\S 3"'] \arrow[dr, dash, "\S 1", "\text{various classical techniques}"'] & & \text{representation theory} \arrow[dl, dash, "\S 2"', "\text{linked Schubert varieties}"] \\ & \text{linkage} & \end{tikzcd}$$

To set up the discussion precisely, let R be a commutative Noetherian ring. We say that an ideal $I \subset R$ is *perfect* if $c \coloneqq \operatorname{grade}(I)$, the maximal length of an regular sequence contained in I, is equal to the projective dimension $\operatorname{pdim} R/I$. For simplicity of exposition, we assume R is a regular local (or graded) ring, with maximal ideal $\mathfrak{m}$ and residue field k. This is the primary case of interest. In this situation the notions of grade and codimension coincide, and I being perfect is equivalent to R/I being Cohen-Macaulay. In particular, if R is a standard graded polynomial ring, this amounts to the study of arithmetically Cohen-Macaulay subschemes of projective space.

1. Past results in the field

There has been an extensive amount of work analyzing the structure of perfect ideals $I \subset R$. For $c = 1$ the problem is trivial, as the ideal of a hypersurface is generated by a single equation. The first major result in this area dates back to the 1890s, when Hilbert proved a structure theorem for ideals I in a polynomial ring with $\operatorname{pdim} R/I = 2$ [8]. This was generalized by Burch in 1968 to arbitrary commutative rings [3]. Using the Hilbert-Burch theorem, one concludes that if $c = 2$, the ideal I is generated by the $(n-1) \times (n-1)$ minors of a $n \times (n-1)$ matrix, where $n = \mu(I)$ is the minimal number of generators of I.

The situation becomes more mysterious for $c \geq 3$, and many authors throughout the late 20th century have addressed various cases which we now briefly survey. It is helpful to introduce two numerical quantities to assist in organizing the story:

- The *deviation* d of I is the quantity $n - c$. Note that $n \geq c$ always, with equality iff I is a complete intersection. Hence the deviation is a measurement of how far an ideal is from being a complete intersection.
- The *type* t of I is the minimal number of generators of the canonical module $\operatorname{Ext}^c_R(R/I, R)$ of R/I. It is equal to the last Betti number b_c for a minimal free resolution of R/I.

The ring R/I is Gorenstein exactly when $t = 1$, and we also refer to the ideal I itself as being Gorenstein in this case. In 1977, Buchsbaum and Eisenbud characterized Gorenstein ideals of codimension $c = 3$ [2]. Explicitly, I is generated by the $(n-1) \times (n-1)$ pfaffians of a $n \times n$ skew matrix. Here the deviation $d = n - 3$ is necessarily even; the impossibility of odd d was established by Watanabe a few years prior in 1973 using linkage [14]. Also using linkage, one can also settle the case $c = 3$, $d = 1$. All values of $t \geq 2$ are possible here, although the description of I is slightly different for t even versus t odd.

There is no analogous uniform characterization of Gorenstein ideals with $c \geq 4$, but certain special cases have been studied. In 1974, Kunz proved that $d = t = 1$ is impossible for arbitrary c, i.e. that Gorenstein ideals can never have deviation 1 [10]. It was surprisingly difficult to come up with nontrivial examples of Gorenstein ideals of deviation 2. For $(c, d, t) = (4, 2, 1)$, Herzog-Miller (1985) and Vasconcelos-Villarreal (1986) proved under mild hypotheses that any such I must be a hypersurface section of the case $(c, d, t) = (3, 2, 1)$ already discussed [7], [13].

But also in 1985, Huneke and Ulrich produced a new interesting family of Gorenstein ideals of odd codimension $c \geq 5$ with $d = 2$ [9]. Their construction also makes sense for $c = 3$, but it coincides with the Buchsbaum-Eisenbud example in that case. In 1988, Lopez showed that if $(c, d, t) = (5, 2, 1)$ and I is in the linkage class of a complete intersection (licci), then it must be either a double hypersurface section of an ideal with $(c, d, t) = (3, 2, 1)$, or a specialization of the $c = 5$ Huneke-Ulrich example [11]. He also produced a new family of Gorenstein ideals of even codimension $c \geq 6$, $t = 2$.

2. A unified perspective

The list of results given in §1 is by no means exhaustive, but progress in this area has slowed down since this flurry of activity in the 70s and 80s. Indeed, the mixed bag of classification and non-existence results presented above does not suggest any obvious patterns, only that the situation becomes more complicated as the values of c, d, t increase. Moreover, all families of perfect ideals described above are licci. There exist non-licci perfect ideals with $c \geq 3$, but their deformation theory is significantly more complicated. Their study is undoubtedly important as well, but one does not expect clean classification results akin to the ones surveyed above in this greater generality.

Restricting our attention to the licci case, we have the following natural questions.

(1) Can we classify all licci ideals with a given grade c, deviation d, and type t?
(2) All licci ideals are perfect. When (in terms of c, d, t) can we guarantee that perfect ideals are licci?

Assuming equicharacteristic zero, there is a conjectural answer to both questions, which we[1] can prove for $c = 3$ [5]. Given the seemingly disparate examples discussed above, it is perhaps surprising that an answer—even if conjectural—exists at all! It reveals a pattern to the apparent chaos, unifying all of the aforementioned examples in a single framework. It also reveals a path forward into previously inaccessible territory. For example, there are exactly 90 families of perfect ideals with $(c, d, t) = (3, 2, 4)$, a case which would've been intractable with previously existing tools.

The likely reason that this pattern has been previously overlooked is that it originates from a rather unexpected source. Some heavy setup is needed to state it; we will clarify the situation with some examples afterwards.

[1]This is joint work with Jerzy Weyman and Lorenzo Guerrieri.

Conjecture 1 (The linkage class of a complete intersection)**.** *Fix integers $c \geq 2$, $d \geq 0$, and $t \geq 1$. Let T denote the graph*

$$\begin{array}{ccccccccccccc} x_{c-2} & — & \cdots & — & x_1 & — & u & — & y_1 & — & \cdots & — & y_d \\ & & & & & & | & & & & & & \\ & & & & & & z_1 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & z_t & & & & & & \end{array}$$

To this graph there is an associated Kac-Moody Lie algebra $\mathfrak{g}$ and group G. If T is a Dynkin diagram, then these are finite-dimensional.

Let $P^+_{x_{c-2}}$ be the maximal positive parabolic associated to the vertex x_{c-2}, and $P^-_{z_1}$ the maximal negative parabolic associated to z_1. The homogeneous space $G/P^+_{x_{c-2}}$ can be viewed as a projective (ind-)variety, inside of which there is a Schubert (ind-)variety X^w of codimension c that is complementary to the open $P^-_{z_1}$-orbit. Explicitly w is the element $s_{z_1}s_u s_{x_1}\cdots s_{x_{c-2}}$ of the Weyl group W. For every $p \in X^w$, the local defining ideal of X^w in $G/P^+_{x_{c-2}}$ at p is perfect of grade c, deviation $\leq d$, and type $\leq t$.

1. *Let $I \subset R$ be licci of grade c, deviation $\leq d$, and type $\leq t$. Then there exists a map* $\operatorname{Spec} R \to G/P^+_{x_{c-2}}$ *pulling back the defining equations of X^w to generators of I. Let k denote the residue field of R. Then the $P^-_{z_1}$-orbit containing the resulting k-point of X^w is independent of the map chosen. Thus $P^-_{z_1}$-orbits in X^w are in bijection with families of such ideals I.*
2. *(Licci conjecture) All perfect ideals of grade c, deviation d, and type t are licci if and only if T is a Dynkin diagram.*

The setup of the conjecture is somewhat imprecise for the non-Dynkin cases, but we avoid discussing technical details here. Assuming T is Dynkin, we can show the existence of the map $\operatorname{Spec} R \to G/P^+_{x_{c-2}}$ in (1), exploiting the fact that X^w is geometrically linked to another Schubert variety. Even without knowing uniqueness of the $P^-_{z_1}$-orbit claimed in (1), we can already conclude that there are finitely many families of licci ideals with the corresponding grade, deviation, and type, simply because there are finitely many orbits. These orbits are in correspondence with certain double cosets of the Weyl group, and thus they may be algorithmically enumerated.

Here is a sketch of how this conjecture recovers the assortment of classification results stated previously.

- $(c,d,t) = (2, n-2, n-1)$. The graph is A_{2n-2} and the homogeneous space is $Gr(n-1, 2n-1)$. Representing points of this space by $(n-1)\times(2n-1)$ matrices, the $P^-_{z_1}$-orbits are determined by the rank of a particular $(n-1)\times n$ block. The open orbit is where this rank is maximal, and the complement X^w is thus cut out by the $(n-1)\times(n-1)$ minors of the block, recovering the generic Hilbert-Burch ideal. It is comprised of $n-1$ orbits, corresponding to all possibilities for the deviation between 0 and $n-2$.
- $(c,d,t) = (3, n-3, 1)$. The graph is D_n. The homogeneous space is the orthogonal Grassmannian $OG(n, 2n)$. The Schubert variety X^w has $\lfloor \frac{n-1}{2} \rfloor$ orbits. Notice that as a function of n, this increases by 1 only when n is odd. This reflects Watanabe's result that there are no grade 3 Gorenstein ideals minimally generated by an even number of elements. The defining equations of X^w along each orbit recovers a generic Buchsbaum-Eisenbud example.
- $(c,d,t) = (n-1, 1, 1)$. The graph is D_n again, but this time the homogeneous space is just a smooth quadric hypersurface in $\mathbb{P}^{2n-1}$, namely the vanishing locus of the quadratic form on the standard representation of $SO(2n)$. There is only a single $P^-_{z_1}$-orbit in X^w. But $(c,d,t) =$

$(n-1,0,1)$ already accounts for this orbit, i.e. it corresponds to a complete intersection. So the conjecture correctly predicts Kunz's result that there are no Gorenstein ideals of deviation 1.

- $(c,d,t)=(c,2,1)$, with $c \geq 3$. The graph is E_{c+2} (with $E_5 = D_5$). The deceptively simple structure of Gorenstein ideals of deviation 2 witnessed in low codimension can be attributed to the fact that $G/P_{x_{c-2}}$ is a relatively "small" homogeneous space for E_{c+2}. In particular, X^w has only a few orbits. The case $c=3$ has already been discussed: there are two orbits, corresponding to complete intersections and ideals generated by the 4×4 pfaffians of a 5×5 skew matrix. Increasing c to 4 does not increase the number of orbits, reflecting the results of Herzog-Miller and Vasconcelos-Villarreal. But for $c=5$ an additional orbit appears, yielding the first Huneke-Ulrich example. Increasing to $c=6$ introduces *three* new orbits, one of which was described by Lopez.
- The previous examples have resulted in relatively few orbits. For contrast, consider $(c,d,t)=(3,2,4)$. The graph is E_8, and the Schubert variety X^w has 108 orbits. Of these, 18 are already accounted for by smaller d or t. As Conjecture 1 is known to hold for $c=3$, we conclude that there are 90 families of grade 3 perfect ideals with deviation 2 and type 4.

3. The case $c = 3$

The condition that $I \subset R$ is a perfect ideal can be restated purely in terms of free resolutions: if $\mathbb{F}$ is a minimal free resolution of R/I, then its dual $\mathbb{F}^*$ is also acyclic (resolving the canonical module). Returning to the well-understood case $c=2$, recall that the classification of perfect ideals comes as a corollary of the Hilbert-Burch theorem, which concerns free resolutions of length 2. Given this, it should not be surprising that the machinery used to prove Conjecture 1 for $c=3$ comes from the structure theory of free resolutions of length 3.

More precisely, the Hilbert-Burch theorem describes the "generic example" of a free resolution

$$0 \to R^{n-1} \to R^n \to R.$$

Weyman constructed generic free resolutions of length 3 in 1989, which are significantly more complicated [16]. Some details were unresolved in the original paper, and the structure of the generic resolution remained opaque until 2018, when a deep connection to the Kac-Moody Lie algebra mentioned in Conjecture 1 resolved the missing details and greatly elucidated the structure of the generic examples [15]. Ongoing work on this project has gradually demystified the appearance of this Lie algebra, and the current formulation of Conjecture 1 and its resolution for $c=3$ using Weyman's construction marks a triumph of the theory [5]. This connection also enables us to describe free resolutions of all grade 3 licci ideals using representation theory, vastly generalizing the resolutions known for the familiar $t=1$ and $d=1$ cases [6]. Some cases of the free resolution are explicitly described in [12].

4. Future directions

There are endless possibilities for future work, an obvious one being the resolution of Conjecture 1 in general.

- An offshoot of (1) is a conjectural classification of all rigid licci algebras. Given a $P_{z_1}^-$-orbit Y, let C_σ be the smallest finite-dimensional Schubert cell meeting the orbit. The completion of the local ring of $X^w \cap C_\sigma$ at a point $p \in Y \cap C_\sigma$ is conjecturally a rigid licci algebra R_σ, and we expect all such algebras to arise in this manner. It is possible to read off data such as the dimension, embedding codimension, deviation, and type of R_σ from combinatorics of

$\sigma \in W$. This conjecture has been confirmed to match an exhaustive list up to dimension 12 proven by Ulrich.

- For the "if" part of (2), there are only finitely many remaining cases to consider, all coming from Dynkin diagrams E_n. As $c = 3$ is settled, the remaining cases either have $d = 1$ or $t = 1$. It is sufficient to consider one of these two cases because they are linked, so we focus on $t = 1$.

  If the arm of length 2 is the left arm, then we get Gorenstein ideals of grade 4 with deviation ≤ 4. In this setting there is a partial analogue of Weyman's theory of generic free resolutions, so with more work the proof of the $c = 3$ licci conjecture might carry over.

  Otherwise, the arm of length 2 is the right arm, corresponding to the remaining two cases of Gorenstein ideals of grade 5 or 6 with deviation 2.
- At one point it was conjectured that Gorenstein ideals of deviation 2 are all licci, perhaps because so few examples were known. However, the "only if" part of (2) makes a different prediction: non-licci ideals with $d = 2$ and $t = 1$ should appear when $c \geq 7$. It is not clear how one might produce such an example.
- More generally, it remains a tantalizing question whether this theory can be extended to explain the behavior of non-licci perfect ideals, examples of which abound for $c \geq 3$. There are particularly simple examples with $(c, d, t) = (3, 3, 3)$, $(3, 5, 2)$, $(3, 2, 5)$, or $(4, 5, 1)$. Examples of the first three are given in [4], and the prototypical "Tom and Jerry" examples of [1] belong to the last case. The T-shaped graph in each of these instances is an *affine* Dynkin diagram. The representation theory of the associated Lie algebra is an enormous industry, and a simple starting question is whether it can be used to produce these examples of non-licci perfect ideals.

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